THEORY OF STABILITY OF COMPRESSIBLE ELASTOPLASTIC GROUND

A. N. Sporykhin

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Based on the three-dimensional linearized equations of stability, the deformation process of compressible elastoplastic ground is investigated in the case of small subcritical deformations. In the case of a homogeneous subcritical state, the general solutions of the equations of stability are constructed similarly to [1].

We shall consider a compressible elastoplastic ground, for which we shall determine the physical equations following [2]. We shall suppose that the condition of the limiting state of the material is represented in the form

$$\Phi(\sigma, \Sigma_2, \Sigma_3) = 0, \tag{1}$$

where $\sigma$ is the first invariant of the stress tensor; $\Sigma_2$ and $\Sigma_3$ are the second and third invariants of the deviator of the stress tensor. Suppose that for this

$$e_{ij} \doteq e_{ij}^{e} + e_{ij}^{p} \tag{2}$$

($e_{ij}$ are the components of the deformation tensors), and that the elastic deformations are related to the stresses by Hooke's law

$$e_{ij}^{e} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}. \tag{3}$$

We shall take the expressions which define the relation between the tensor of the rates of the plastic deformations and the stresses in the form [2]

$$\varepsilon_{ij}^{p} = \lambda\left[\frac{1}{3}\frac{\partial\Phi}{\partial\sigma}\delta_{ij} + \frac{\partial\Phi}{\partial\Sigma_2}\frac{\partial\Sigma_2}{\partial\sigma_{ij}} + \frac{\partial\Phi}{\partial\Sigma_3}\frac{\partial\Sigma_3}{\partial\sigma_{ij}}\right] + \psi(\sigma)\dot{\sigma}\delta_{ij}. \tag{4}$$

Here $\sigma = {}^1/_3\, d\sigma_{kk}/dt$; $\psi(\sigma) = d\varphi/d\sigma$; $e = {}^1/_3\, e_{kk}$; $\varphi(\sigma) - e = 0$ is a function of the bulk loading, which is determined completely from experiments by omnidirectional uniform tension–compression; $\lambda \geq 0$ is an indeterminate factor.

The total deformations are related to the displacements by the Cauchy formulas

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{5}$$

The equations for equilibrium in the absence of body forces and the boundary conditions with specified forces at the surface of the body will be taken in the form [3]

$$[\sigma_{jk}(\delta_{ik} + u_{i,k})]_{,j} = 0, \quad [\sigma_{jk}(\delta_{ik} + u_{i,k})]n_j = P_i, \tag{6}$$

where $n_j$ are unit vectors of the normal to the surface of the body, $P_i$ are the components of the surface forces.

Suppose that the solution of the system of equations (1)-(6) is

$$\sigma_{ij}^{0}(x_k, t),\ e_{ij}^{0}(x_k, t),\ e_{ij}^{0p}(x_k, t),\ u_i^{0}(x_k, t), \ldots.$$

Subsequently, the stability of this process in relation to small perturbations will be investigated.

We shall represent the quantities associated with the perturbed form of motion in the form

$$\sigma_{ij} = \sigma_{ij}^{0} + \sigma_{ij}^{+},\ e_{ij} = e_{ij}^{0} + e_{ij}^{+},\ e_{ij}^{p} = e_{ij}^{0p} + e_{ij}^{+p}, \ldots.$$

The components of the characteristics of the perturbed motion are not marked with any indices but the perturbations are marked with the superscript +. Then expansion of Eq. (1) with an

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accuracy up to linear terms gives

$$\left(\frac{\partial\Phi}{\partial\sigma}\right)_0\sigma^+ + \left(\frac{\partial\Phi}{\partial\Sigma_2}\right)_0\Sigma_2^+ + \left(\frac{\partial\Phi}{\partial\Sigma_3}\right)_0\Sigma_3^+ = 0,$$

where

$$3\sigma^+ = \sigma_{kk}^+;\ \Sigma_2^+ = 2s_{ij}^0 s_{ij}^+;\ \Sigma_3^+ = 3s_{ij}^+ s_{jp}^0 s_{pi}^0. \tag{7}$$

From relations (2) and (3), we have

$$e_{ij}^+ = e_{ij}^{+e} + e_{ij}^{+p},\ e_{ij}^{+e} = \frac{1+\nu}{E}\sigma_{ij}^+ - \frac{\nu}{E}\sigma_{kk}^+\delta_{ij}. \tag{8}$$

The associated law of flow (4) assumes the form

$$\varepsilon_{ij}^{+p} = \lambda_0\xi_{ij}^+ + \lambda^+\frac{\partial\Phi^0}{\partial\sigma_{ij}^0} + \eta^+\delta_{ij}, \tag{9}$$

where

$$\xi_{ij}^+ = \frac{1}{3}\left\{\frac{\partial^2\Phi}{\partial\sigma^2}\sigma^+ + \left[2\frac{\partial^2\Phi}{\partial\sigma\partial\Sigma_2}s_{kl}^0 + 3\frac{\partial^2\Phi}{\partial\sigma\partial\Sigma_3}p_{lk}^0\right]s_{kl}^+\right\}\delta_{ij} + 2\left\{\frac{\partial^2\Phi}{\partial\Sigma_2\partial\sigma}\sigma^+ + \left[2\frac{\partial^2\Phi}{\partial\Sigma_2^2}s_{kl}^0 + 3\frac{\partial^2\Phi}{\partial\Sigma_2\partial\Sigma_3}p_{lk}^0\right]s_{kl}^+\right\}s_{ij}^0 + 2\frac{\partial\Phi}{\partial\Sigma_2}s_{ij}^+ +$$

$$+ 3\left\{\frac{\partial^2\Phi}{\partial\Sigma_3\partial\sigma}\sigma^+ + \left[2\frac{\partial^2\Phi}{\partial\Sigma_3\partial\Sigma_2}s_{kl}^0 + 3\frac{\partial^2\Phi}{\partial\Sigma_3^2}p_{lk}^0\right]s_{kl}^+\right\}p_{ij}^0 + 3\frac{\partial\Phi}{\partial\Sigma_3}s_{pi}^0 s_{jp}^+;\ p_{lk}^0 = s_{cp}^0 s_{pk}^0;$$

$$\lambda_0 = \left[\delta\left(\varepsilon_{ij}^{0p}\varepsilon_{ij}^{0p} - \frac{1}{3}\varepsilon_{kk}^{0p}\varepsilon_{kk}^{0p}\right)\right]^{1/2};\quad \lambda^+ = \delta\left[\varepsilon_{mn}^{+p} - \frac{1}{3}\varepsilon_{kk}^{+p}\delta_{mn} - \lambda_0\left(\xi_{mn}^+ - \frac{1}{3}\xi_{kk}^+\delta_{mn}\right)\right]\frac{\partial\Phi^0}{\partial\sigma_{mn}^0}; \tag{10}$$

$$\delta = \left[\frac{\partial\Phi^0}{\partial\sigma_{mn}^0}\frac{\partial\Phi^0}{\partial\sigma_{mn}^0} - \frac{1}{3}\left(\frac{\partial\Phi^0}{\partial\sigma^0}\right)^2\right]^{-1};\ \eta^+ = \psi(\sigma_0)\dot{\sigma}^+ + \left(\frac{\partial\psi}{\partial\sigma}\right)_0\dot{\sigma}^0\sigma^+$$

For $e_{ij}^+$ we obtain

$$2e_{ij}^+ = u_{i,j}^+ + u_{j,i}^+. \tag{11}$$

The linearized equation of equilibrium and the boundary conditions have the form [3, 4]

$$(\sigma_{ij}^+ + \sigma_{jk}^0 u_{i,k}^+)_{,j} - \rho\ddot{u}_i^+ = 0,\quad (\sigma_{ij}^+ + \sigma_{jk}^0 u_{i,k}^+)n_j = P_i^+. \tag{12}$$

The boundary problem (7)-(12), in a similar way to the method in [5], can be reduced to the investigation of a system of differential equations with constant coefficients and, thus, we shall carry out the investigation of the stability of system (7)-(12) according to the limiting system of equations.

In this case, Eq. (9) assumes the form

$$\varepsilon_{ij}^{+p} = \lambda^+\frac{\partial\Phi^0}{\partial\sigma_{ij}^0} + \eta^+\delta_{ij}, \tag{13}$$

so that

$$\eta^+ = \psi(\sigma_0)\dot{\sigma}^+,\ \lambda^+ = \delta\left(\varepsilon_{mn}^{+p} - \frac{1}{3}\varepsilon_{kk}^{+p}\delta_{mn}\right)\frac{\partial\Phi^0}{\partial\sigma_{mn}^0}. \tag{14}$$

The other relations of the system of equations (7)-(12) retain the previous form with the only difference being that quantities with index zero occurring in them are certain stationary quantities. It is obvious that the results obtained above can be extended to the case when the stress function is chosen in more general form [6]:

$$\Phi(\sigma, \Sigma_2, \Sigma_3, e^p, E_2^p, E_3^p, \Pi_2, \Pi_{12}, \Pi_{21}, k_i) = 0.$$

For determinacy, we choose the condition of plasticity

$$\Phi = \alpha\sigma^0 + \sqrt{\frac{1}{2}}\Sigma_2^{0\,1/2} - k = 0, \tag{15}$$

which is characteristic for friable media. Equation (7) in this case is written in the form

$$2\alpha(k - \alpha\sigma^0)\sigma^+ + s_{ij}^0 s_{ij}^+ = 0. \tag{16}$$

Equations (13), taking account of Eq. (14) for the condition of plasticity (15), assume the form

$$\varepsilon_{ij}^{+p} = \frac{s_{mn}^0}{k - \alpha\sigma^0}\left(\varepsilon_{mn}^{+p} - \frac{1}{3}\varepsilon_{kk}^{+p}\delta_{mn}\right)\left[\frac{s_{ij}^0}{2(k - \alpha\sigma^0)} + \frac{\alpha}{3}\delta_{ij}\right] + \psi(\sigma_0)\dot{\sigma}^+\delta_{ij}. \tag{17}$$

Eliminating from relation (8), (16), and (17) the quantities $e_{ij}^{+e}$, $e_{ij}^{+p}$, and $\sigma^+$, we can obtain after a series of transformations

$$\frac{1+\nu}{E}\sigma_{ij}^{+} = e_{ij}^{+} - \left(as_{ij}^0 + \frac{1}{3}\delta_{ij}\right)e_{mn}^{+} + c\left[a\left(b - 3\psi(\sigma_0)\right)s_{ij}^0 - \frac{1+\nu}{E}\delta_{ij}\right]\left[2\alpha^2 as_{kl}^0 e_{kl}^{+} - e_{mn}^{+}\right], \tag{18}$$

where

$$a = 2\alpha\,(k - \alpha\sigma^0);\; b = 3(2\nu - 1)E^{-1};\; c = E\left[2\alpha^2(1+\nu) - E(b - 3\psi(\sigma_0))\right]^{-1}.$$

Let us consider the case of the principal stressed state in the form

$$\sigma_{ii}^0 = \mathrm{const}_i, \quad \sigma_{ij}^0 = 0, \quad i \neq j.$$

The linearized three-dimensional equations of motion (12) in this case can be represented in the form

$$\{\sigma_{ij} - [q(\delta_{j1}u_{i,1} + \delta_{j2}u_{i,2}) + p\delta_{i3}u_{i,3}]\}_{,j} - \rho s^2 u_i = 0, \tag{19}$$

and the boundary conditions in the form

$$\{\sigma_{ij} - [q(\delta_{j1}u_{i,1} + \delta_{j2}u_{i,2}) + p\delta_{j3}u_{i,3}]\}n_j = P_i. \tag{20}$$

Here and in the future, in the components of vectors and tensors characterizing perturbations, the time factor exp st will be excluded, and for the amplitude quantities of the perturbations, the index + will be omitted. In the equations (19) and (20), it is assumed that the elastoplastic body is compressed along the axis $Ox_3$ by forces of intensity p and along the axis $Ox_1$ $Ox_2$ by forces of intensity q.

The linearized relation (18) between the stresses and deformations for compressed elastoplastic ground, for the case of the principal motion described above, can be represented in the form

$$\begin{gathered}\sigma_{ij} = \delta_{ij}a_{jk}u_{k,k} + (1 - \delta_{ij})G_{ij}(u_{i,j} + u_{j,i}),\ (\Sigma k),\\ a_{jl} = \frac{E}{1+\nu}\delta_{jl} + \left(2\alpha^2 as_{ll}^0 - 1\right)B_{jj} - A_{jj},\quad G_{ij} = \frac{E}{2(1+\nu)},\\ A_{ij} = \frac{E}{1+\nu}\left(as_{ij}^0 + \frac{1}{3}\delta_{ij}\right),\quad B_{ij} = \frac{cE}{1+\nu}\left[a\left(b - 3\psi(\sigma_0)\right)s_{ij}^0 - \frac{1+\nu}{E}\delta_{ij}\right].\end{gathered} \tag{21}$$

It is obvious that

$$a_{11} = a_{22},\quad a_{13} = a_{23},\quad G_{ij} = \frac{E}{2(1+\nu)} = G.$$

Consequently, expressions (21) can be considered as the Hooke's law relations for a transcendental-isotropic body, in which the plasma of isotropy coincides with the plane $x_1Ox_2$.

Substitution of expressions (21) in Eq. (19) leads to a system of equations in amplitude displacements

$$L_{ij}u_j = 0. \tag{22}$$

The differential operators $L_{ij}$ have the form

$$\begin{gathered}L_{ij} = \delta_{ij}\left(M_{in}\frac{\partial^2}{\partial x_n^2} - \rho s^2\right) + (1 - \delta_{ij})F_{ij}\frac{\partial^2}{\partial x^i\partial x^j},\quad \left(\textstyle\sum_{i,j};\sum_n\right);\\ M_{in} = \{a_{ii} - q(\delta_{1n} - \delta_{2n}) - p\delta_{3n}\ (i = n);\ G_{in} - q(\delta_{1n} - \delta_{2n}) - p\delta_{3n}\ (i \neq n)\},\\ F_{ij} = a_{ij} + G_{ij}.\end{gathered}$$

In order to obtain Eq. (22), as in [1], the general solutions in invariant form can be constructed in a similar way.

For a cylindrical body with a curvilinear contour of the transverse section, the general solutions of the stability equations can be written in the form

$$\begin{gathered}u_n = \frac{\partial}{\partial\tau}\Psi - \frac{\partial^2}{\partial n\partial x_3}\chi,\quad u_\tau = -\frac{\partial}{\partial n}\Psi - \frac{\partial^2}{\partial\tau\partial x_3}\chi,\\ u_3 = \frac{a_{11}}{F_{23}}\left(\Delta + \frac{G}{a_{11}}\frac{\partial^2}{\partial x_3^2} - \frac{\rho s^2}{a_{11}}\right)\chi,\quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2},\end{gathered} \tag{23}$$

where n and $\tau$ are the normal and tangent to the contour of the transverse section.

The functions $\Psi$ and $\chi$ are determined from the equations

$$\left(\Delta + \frac{\partial^2}{\partial x_3^2} - \frac{\rho s^2}{G}\right)\Psi = 0,\quad \left[\left(a_{11}\Delta + G\frac{\partial^2}{\partial x_3^2} - \rho s^2\right)\left(G\Delta + a_{33}\frac{\partial^2}{\partial x_3^2} - \rho s^2\right) - F_{13}F_{31}\Delta\frac{\partial^2}{\partial x_3^2}\right]\chi = 0. \tag{24}$$

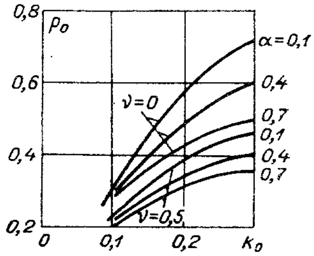


Fig. 1

In quasistatic formulation (s = 0), the functions $\Psi$ and $\chi$ are solutions of the equations

$$\left(\Delta+\frac{\partial^2}{\partial x_3^2}\right)\Psi=0,\quad \left[\Delta^2+\left(\xi_2^2+\xi_3^2\right)\Delta\frac{\partial^2}{\partial x_3^2}+\xi_2^2\xi_3^2\frac{\partial^4}{\partial x_3^4}\right]\chi=0, \tag{25}$$

where the constants $\xi_i^2$ have the form

$$\xi_{2,3}^2=\frac{a_{11}a_{33}+G^2-F_{13}F_{31}}{2a_{11}G}\pm\left[\left(\frac{a_{11}a_{33}+G^2-F_{13}F_{31}}{2a_{11}G}\right)^2-\frac{a_{33}}{a_{11}}\right]^{1/2}.$$

If in relations (23)-(25) we suppose that for $a_{ij}$ in expression (21) $\alpha$ = 0 and $c=b^{-1}$ [$\alpha=0$ and $\psi(\sigma_0)$ = 0], then we arrive at the results of [1].

The solutions derived by analogy with the results obtained for elastic, viscoelactic, elastoplastic [1], and elastoviscoplastic bodies [7] for small uniform subcritical deformations, allow us to obtain the characteristic determinants for a number of problems.

Thus, in the case of a plate, infinitely long in the direction $Ox_1$, with thickness 2h and length $l$ when compressed along the axis $Ox_3$ by a "dead" load of intensity p, we obtain the characteristic determinant in the usual way in the form

$$\begin{aligned}&\left(\xi_1^2\xi_2^2-\frac{a_{33}}{a_{32}}\frac{a_{33}^2-a_{32}^2-a_{32}G}{a_{22}G}\right)(\xi_1\operatorname{sh}\alpha\xi_1\operatorname{ch}\alpha\xi_2-\xi_2\operatorname{sh}\alpha\xi_2\operatorname{ch}\alpha\xi_1)-\\&-\frac{a_{33}^2-a_{32}^2-a_{32}G}{a_{22}G}\xi_1\xi_2(\xi_2\operatorname{sh}\alpha\xi_1\operatorname{ch}\alpha\xi_2-\xi_1\operatorname{sh}\alpha\xi_2\operatorname{ch}\alpha\xi_1)+\\&+\frac{a_{33}}{a_{32}}\left(\xi_1^3\operatorname{sh}\alpha\xi_1\operatorname{ch}\alpha\xi_2-\xi_2^3\operatorname{sh}\alpha\xi_2\operatorname{ch}\alpha\xi_1\right)=0,\quad \alpha=\frac{\pi h}{l}.\end{aligned} \tag{26}$$

The solution obtained can be used [8] in order to determine the stable dimensions of extended (strip) barriers in a compressed elastoplastic ground massif. In this case, $p = \gamma HL(2h)^{-1}$, where $\gamma$ is the weight by volume of the rock; H is the distance from the ground surface to the roof of the chamber, and L is the base of the column of rock pressing on the barrier.

In the case of surface instability with the condition that the loss of stability occurs within the bounds of plane deformation in the plane $x_3Ox_2$, the characteristic equation has the form

$$(\xi_1-\xi_2)\left[\xi_1^2\xi_2^2+\frac{a_{33}}{a_{32}}\left(\xi_1^2+\xi_2^2\right)+\xi_1\xi_2\left(\frac{a_{22}a_{33}-a_{32}^2-a_{32}G}{a_{22}G}+\frac{a_{33}}{a_{32}}\right)-\frac{a_{33}}{a_{32}}\frac{a_{22}a_{33}-a_{32}^2-a_{32}G}{a_{22}G}\right]=0. \tag{27}$$

Constants $\xi_i^2$ in (26) and (27) have the form

$$\xi_{1,2}^2=\frac{a_{33}^2-(a_{32}+G)^2+(G-p)G}{2a_{22}G}\pm\left\{\left[\frac{a_{33}^2-(a_{32}+G)^2+(G-p)G}{2a_{22}G}\right]^2-\frac{a_{33}(G-p)}{a_{22}G}\right\}^{1/2}.$$

Equation (27) has been solved numerically for different values of $k_0$, $\alpha$, $\nu$, where $k_0 = kE^{-1}$ is the yield point; $\alpha$ is the rate of "dilatancy" ($\alpha = \tan\rho$, where $\rho$ is the angle of internal friction, in particular for sand $\rho$ = 26-40°, hence $\alpha$ = 0.49-0.82); $\nu$ is Poisson's coefficient; E is the modulus elasticity. In order to determine the function defining the volume compression $\psi(\sigma_0)$, the relation between $\sigma$ and $\varepsilon$ was chosen to be linear, $\sigma = \sigma_{max}\varepsilon$, which, according to [9], is characteristic for friable media, in particular for sand. Figure 1 shows the dependence of the critical pressure $p_0 = pE^{-1}$ on the yield point $k_0$ for values of Poisson's ratio of $\nu$ = 0, 0.5 and rate of dilatancy $\alpha$ = 0.1, 0.4, and 0.7, characteristic for friable media (sand, gravel, etc).

A calculation has shown that the effect of $\nu$ and $\alpha$ within the above-stated limits on the magnitude of the critical force is significant. However, the arbitrary values of the critical loadings obtained in this case are unreal and, consequently, no surface instability is observed in practice.

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# VARIATIONAL METHOD FOR SOLVING PROBLEMS OF THE PLASTICITY OF COMPRESSIBLE MEDIA


I. S. Degtyarev



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Variational methods used in the theory of plastic flow are formulated on the assumption of the incompressibility of the deformable medium. In solving problems of the mechanics of soils and friable media and technological problems of the plastic shaping of uncompacted materials it is very important to take account of irreversible volumetric change. Extremum and variational theorems are proved in [1, 2] for rigid-plastic and viscoplastic expanding bodies. A variational equation equivalent to a complete system of differential equations is derived for a compressible plastic body.


We consider a material medium with the equations of state

$$S_{ij} = 2g_1(\sigma, H)\,\varepsilon^*_{ij}, \quad \rho = \varphi(\sigma), \quad \varepsilon^*_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon\delta_{ij}, \tag{1}$$

where the $S_{ij}$ and the $\varepsilon^*_{ij}$ are, respectively, the components of the stress deviators and the strain rates; $g_1(\sigma, H)$ and $\varphi(\sigma)$ are functions of the material; $\rho$ is the density of the medium; H is the intensity of shear strain rates; and $\sigma$ is the mean stress.


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